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The conformal anomaly for the non-relativistic Landau problem

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Abstract. Using the method due to Fujikawa wherein anomalies are identified with the Jacobian factors arising from the path integral measure under symmetry transformations, the anomaly under translations of inverse time, i.e. $1/t \to 1/t - a$ is worked out for the Landau problem.

The symmetries associated with the planar motion of charged particles under the influence of various electromagnetic potentials has been the subject of several recent papers [1–5]. Among the various generators of these symmetries, it was found [1–5] that the generators of three time-reparametrization transformations, namely time translation ($t \to t + t_0$), time dilatation ($t \to e^{-\rho} t$) and translation of inverse time ($1/t \to 1/t - a$), close on commutation to the conformal $SO(2, 1)$ Lie algebra. The Landau problem also concerns the planar motion of a charged point particle, but in the presence of a uniform magnetic field $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$, $2\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r}$. However, it turns out that the Landau Lagrangian, quite unlike the other cases of planar motion discussed in [1–5] is non-invariant [6] under $t \to e^{-\rho} t$ and $1/t \to 1/t - a$.

The recent observation by Jackiw [7] that the aforesaid $SO(2, 1)$ symmetry does not survive quantization in the case of planar motion in a two-dimensional $\delta^{(2)}(\boldsymbol{r})$ potential motivated us to ask if the Landau problem was afflicted with a dilatation anomaly. An affirmative answer was obtained recently [6] from an analysis of the naive Ward identity associated with broken dilatation symmetry. This paper extends this recent calculation [6] to derive the conformal anomaly, i.e. the anomaly associated with the transformation $1/t \to 1/t - a$. The method we adopt is that due to Fujikawa [8, 9] wherein the anomalies are identified with the Jacobian factors arising from the path integral measure under the symmetry transformations. Apart from the apparent simplicity of the Fujikawa approach [10] in contrast to the somewhat laborious verification of the naive Ward identity adopted earlier [6], its use in the present paper anticipates similar calculations in non-relativistic field theories that will be reported elsewhere.

We begin with the generating functional $Z[\boldsymbol{J}]$ of connected Green functions for the Landau Lagrangian, namely [6]

$$W[\boldsymbol{J}] = \exp \mathrm{i} Z[\boldsymbol{J}] = \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{q}'\, \Phi_0^*(\boldsymbol{q}' t_0')\Phi_0(\boldsymbol{q} t_0)\langle \boldsymbol{q}' t_0' | \boldsymbol{q} t_0\rangle^{J} \tag{1}$$

with

$$\langle \boldsymbol{q}' t_0' | \boldsymbol{q} t_0\rangle^{J} = \lim_{n\to\infty} \int \prod_{i=1}^{n} \left[\mathrm{d}\boldsymbol{x}\,(t_i)\, \frac{m}{2\pi \mathrm{i}\varepsilon} \right] \exp \mathrm{i} \int_{t_0}^{t_0'} \mathrm{d}s (L + \boldsymbol{J} \cdot \boldsymbol{x}) \tag{2}$$

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and

$$L = \tfrac{1}{2}m\dot{x}^2 + \frac{e}{c}\,\dot{x}\cdot A.$$

Note that Φ_0 and Φ_0^* denote the ground state wavefunctions in (1) and $t_0' - t_0 = (n+1)\varepsilon$. Under the conformal transformation $t \to t' = t/(1 - ta(t))$, we have following Jackiw [1]

$$\begin{aligned} \delta x_i &= a(t)(tx_i - t^2\dot{x}_i) \\ \delta \dot{x}_i &= \dot{a}(t)(tx_1 - t^2\dot{x}_i) + a(t)(x_i - t\dot{x}_i - t^2\ddot{x}_i) \end{aligned} \tag{3}$$

and

$$\delta L = \dot{a}(t)Q + a\frac{2et}{c}\,\dot{x}\cdot A + \frac{\mathrm{d}}{\mathrm{d}t}\left(-at^2L + \tfrac{1}{2}amx^2\right) \tag{4}$$

with $Q = -t^2H + t(\pi\cdot x) - \frac{1}{2}\mathrm{m}x^2$ $\quad H = \frac{1}{2}mx^2$ and $\pi_i = m\dot{x}_i + (e/c)A_i$. Note the parameter $a(t)$ is now a function of t, so as to derive the Ward identities following the Fujikawa method [8, 9] easily. We now expand

$$x_i(t) = \sum_{k=1}^{\infty} a_{ik}\phi_k(t)$$

with ϕ_k(t) being a complete set of eigenfunctions of the Hermitian operator

$$D_{ij} = \left(-m\delta_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{eB}{c}\,\varepsilon_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\right) \qquad i, j = 1, 2.$$

As a matrix we can write

$$D = \begin{pmatrix} -m\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} & \dfrac{eB}{c}\dfrac{\mathrm{d}}{\mathrm{d}t} \\ -\dfrac{eB}{c}\dfrac{\mathrm{d}}{\mathrm{d}t} & -m\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} \end{pmatrix} = I\left(-m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) + \mathrm{i}\sigma_y\frac{eB}{c}\frac{\mathrm{d}}{\mathrm{d}t} \tag{5}$$

with $D\phi_k = \lambda_k\phi_k$, I being the unit 2×2 matrix and σ_y the usual Pauli matrix. The completeness and orthonormality properties of the eigenfunctions ϕ_k(t) are now defined by the relations

$$\sum_{k=1}^{\infty} \phi_{k,a}(t_1)\phi_{k,b}^*(t_2) = \delta_{ab}\delta(t_1 - t_2) \qquad a, b = 1, 2$$

$$\int \mathrm{d}t\ \phi_k^+\phi_j = \delta_{kj}$$

respectively. Note also that, with the action $S = \int L\,\mathrm{d}t$,

$$\frac{\delta^2 S}{\delta x_k(t_1)\delta x_i(t_2)} = D_{ik}\delta(t_2 - t_1).$$

Using (3) it is easy to see that when $x_i - x'_i(t) = x_i + \delta x_i$

$$a_{ik} \to a'_{ik} = a_{ik} + \sum_j a_{ij} \int \mathrm{d}t\, a(t) \phi_k^+ (t - t^2 \partial_t) \phi_j \tag{6}$$

The change in the path integral measure can now be easily obtained using (6). With the notation $b_k = a_{1k}$ and $c_k = a_{2k}$ we obtain for the measure

$$\begin{aligned} \prod_{i=1}^{n} \mathrm{d}x_1(t_i)\, \mathrm{d}x_2(t_i) &= \prod_{i=1}^{n} \mathrm{d}b_i\, \mathrm{d}c_i \\ &= (\det c_{kl})^{-2} \prod_{i=1}^{n} \mathrm{d}b'_i\, \mathrm{d}c'_i \end{aligned} \tag{7}$$

with $c_{kl} = \delta_{kl} + \int \mathrm{d}t\, a(t) \phi_k^+ (t - t^2 \partial_t) \phi_1$.

On using (7) in (2) we obtain

$$\langle \boldsymbol{q}' t'_0 | \boldsymbol{q} t_0 \rangle^J = \int \prod_{i=1}^{n} \left[m \frac{\mathrm{d}b'_i\, \mathrm{d}c'_i}{2\pi \mathrm{i} \varepsilon} \right] \exp \int_{t_0}^{t'_0} \mathrm{d}t\, (L + \boldsymbol{J} \cdot \boldsymbol{x}) \exp[-2\, \mathrm{Tr} \ln(c_{k1})]. \tag{8}$$

For infinitesmal $a(t)$ the last exponential becomes

$$\exp\left(-2 \sum_k \int \mathrm{d}t\, a(t) \phi_k^+ (t - t^2 \partial_t) \phi_k \right)$$

thus implying that the anomaly is given by (ignoring $a(t)$)

$$A(t) = -2 \sum_k \phi_k^+ (t - t^2 \partial_t) \phi_k(t) \tag{9}$$

Following Fujikawa [8, 9] we shall now use an exponential cut-off to regulate the sum in (9) and write

$$\begin{aligned} A(t) &= -2 \lim_{M \to \infty} \lim_{t \to t'} \sum_k \phi_k^+(t') (t - t^2 \partial_t) \phi_k(t)\, \mathrm{e}^{-\lambda_k^2/M^2} \\ &= -2 \lim_{M \to \infty} \lim_{t \to t'} \mathrm{Tr}\{ (t - t^2 \partial_t)\, \mathrm{e}^{-D^2/M^2}\, \delta(t - t') \} \\ &= -2 \lim_{M \to \infty} \lim_{t \to t'} \mathrm{Tr} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{2\pi} (t - t^2 \partial_t)\, \mathrm{e}^{-D^2/M^2}\, \mathrm{e}^{\mathrm{i}z(t - t')} \end{aligned} \tag{10}$$

where we have now used the completeness of the eigenfunctions $\{\phi_n(t)\}$ and the integral representation of the δ-function in arriving at (10). Using (5) it is easy to see that

$$D^2\, \mathrm{e}^{\mathrm{i}z(t - t')} = \mathrm{e}^{\mathrm{i}z(t - t')} \left[I \left(m^2 z^4 + \left(\frac{eB}{c} \right)^2 z^2 \right) - 2 \frac{eB}{c} m z^3 \sigma_y \right]$$

Thus (10) becomes

$$A(t) = -2 \lim_{M \to \infty} \mathrm{Tr} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{2\pi} (t - \mathrm{i}z\, t^2)\, \mathrm{e}^{-aI + b\sigma_y} \tag{11}$$

with $M^2a=m^2z^4+(eB/c)^2z^2$ and $M^2b=2(eB/c)mz^3$. This can again be converted to

$$A(t)=-4\lim_{M\to\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{2\pi}(t-\mathrm{i}z\,t^2)\,\mathrm{e}^{-a}\cosh b$$

after using the familiar identity $\mathrm{e}^{b\sigma_y}=\mathrm{I}\cosh b+\sigma_y\sinh b$, and completing the trace operation. Since the second term yields zero, we obtain

$$A(t)=-4t\lim_{M\to\infty}\int_0^{\infty}\frac{\mathrm{d}z}{2\pi}\mathrm{e}^{-a}(\mathrm{e}^{b}+\mathrm{e}^{-b}) \tag{12}$$

Substituting for a and b it is easy to check that (12) can be written, after elementary algebraic manipulations, as

$$\pi A(t)=-2t\lim_{M\to\infty}\int_{\omega^2}^{\infty}\mathrm{d}x\,x^{-1/2}\,\mathrm{e}^{-px^2+2qx} \tag{13}$$

with $M^2p=m^2$, $\omega=(eB/2mc)$ and $q=p\omega^2$. We now regard (13) as

$$\pi A(t)=-2t\left\{\lim_{M\to\infty}\int_0^{\infty}\mathrm{d}x\,x^{-1/2}\,\mathrm{e}^{-px^2+2qx}-\lim_{M\to\infty}\int_0^{\omega^2}\mathrm{d}x\,x^{-1/2}\,\mathrm{e}^{-px^2+2qx}\right\}. \tag{14}$$

Two points now need to be made. Firstly, the exchange of the limiting operation with the integration can be made in the second integral to obtain a finite result namely 2ω. The first integral can, on the other hand, be evaluated exactly to obtain [11], with $y\sqrt{2a}=b$

$$\lim_{M\to\infty}(2p)^{-1/4}\Gamma(\tfrac{1}{2})\,\mathrm{e}^{y^2}D_{-1/2}(-y)=\lim_{M\to\infty}(2p)^{-1/4}\,\mathrm{e}^{y^2}(\tfrac{1}{4}y^2)^{1/4}K_{1/4}(\tfrac{1}{4}y^2) \tag{15}$$

where $D_{-1/2}(x)$ and $K_{1/4}(x)$ are the parabolic cylinder functions and modified Bessel functions respectively. Since $K_{1/4}(x)\approx\frac{1}{2}\Gamma(1/4)(x/2)^{-1/4}$ as $x\to 0$, we obtain using (15) the result

$$\pi A(t)=+4t\omega-t\Gamma(1/4)\lim_{M\to\infty}(M/m)^{1/2}. \tag{16}$$

Note that the second term is independent of the charge e; it is therefore the conformal anomaly associated with the free particle Lagrangian $L=\frac{1}{2}m\dot{x}^2$.

With

$$\pi A_f(t)=-t\Gamma(1/4)\lim_{M\to\infty}(M/m)^{1/2}$$

we obtain the correct conformal anomaly for the Landau Lagrangian given by the difference [10]

$$\pi(A-A_f)=4t\omega=4t(eB/2mc). \tag{17}$$

The derivation presented above for the conformal anomaly could of course be carried through for the dilatation transformation $t\to\mathrm{e}^{-\rho}t$; on using $\delta x_i=a(t)(t\partial_t-\frac{1}{2})x_i(t)$ we would obtain an equation analogous to (11) but with $(\mathrm{i}zt-\frac{1}{2})$ in place of $(t-\mathrm{i}z\,t^2)$.

Following the same steps from (11)–(15) one would now obtain in place of (16) the result

$$2\pi A_d(t)=\Gamma(\tfrac{1}{4})\lim_{M\to\infty}(M/m)^{1/2}-4\omega. \tag{18}$$

On identifying the first term with the dilatation anomaly for the free particle, one now obtains the correct dilatation anomaly for the Landau Lagrangian, namely

$$\pi(A_d-A_{d,f})=-2\omega. \tag{19}$$

Thus the conformal anomaly is $-2t$ times the dilatation anomaly; we recall here that the same relation [6] obtained between the time derivatives $\mathrm{d}Q/\mathrm{d}t$ and $\mathrm{d}D/\mathrm{d}t$ of the conformal (Q) and dilatation (D) charges classically. We may also point out that the dilatation anomaly given in (19) is $(2/\pi)$ times the anomaly calculated via the anomalous Ward identity earlier [6]. Indeed, the calculation presented here can also be pursued from the point of view of the naive Ward identity associated with broken conformal symmetry. The conformal anomaly then turns out to be $2t\omega$—which is $(-2t)$ times the dilatation anomaly.

We shall now conclude this paper with a few pertinent remarks. Firstly, it is clear from (10) *et seq.* that if we had chosen a plane wave basis *ab initio* instead of the $\{\phi_k(t)\}$ and used the regulator

$$\exp\left(-I\frac{m^2}{M^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^4\right)$$

then a non-zero conformal anomaly would not have been obtained. Thus the $\phi_k(t)$ basis which diagonalizes the Hermitian operator D should be used to derive exact relations such as Ward identities (cf Fujikawa [8]). Secondly, as also observed by Fujikawa [9], in connection with the trace anomaly, the conformal anomaly (as also the dilatation anomaly) in the present paper arises from the non-commutativity of two basic operators, namely D and the generator W of global conformal transformation $U(\alpha)=\mathrm{e}^{\mathrm{i}\alpha W}$; indeed with

$$U(\alpha)tU^{-1}(\alpha)=(1-\alpha t)^{-1}t$$

and

$$U(\alpha)\frac{\mathrm{d}}{\mathrm{d}t}U^{-1}(\alpha)=(1-\alpha t)^2\frac{\mathrm{d}}{\mathrm{d}t}$$

it is clear that the commutator

$$\mathrm{i}[W,D]=2\,mI\left(2t\frac{\mathrm{d}^2}{\mathrm{d}t^2}-3t\frac{\mathrm{d}}{\mathrm{d}t}\right)-2\,\mathrm{i}\sigma_y\frac{eB}{c}\,t\frac{\mathrm{d}}{\mathrm{d}t}. \tag{20}$$

A similar non-zero result is obtained for global dilatation transformations thus accounting for the dilatation anomaly.

An extension of the work reported herein to (2+1)-dimensional non-relativistic field theories is presently in progress. The details will be reported elsewhere.

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